STRANGE DUALITY AND THE HITCHIN/WZW CONNECTION

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1. Introduction

Let X be a connected smooth projective curve of genus g over $\mathbb{C}$. Assume for simplicity that $g > 2$ (see Section 1.3). Let $SU_X(r)$ be the moduli space of semi-stable vector bundles of rank r with trivial determinant over X. For any line bundle L of degree $g-1$ on X define $\Theta_L = \{E \in SU_X(r), h^0(E \otimes L) \geq 1\}$. This turns out be a non-zero Cartier divisor whose associated line bundle $\mathcal{L} = \mathcal{O}(\Theta_L)$ does not depend upon L. It is known that $\mathcal{L}$ generates the Picard group of $SU_X(r)$ ([DN]).

Let $U^*_X(k)$ be the moduli space of semi-stable rank k and degree $k(g-1)$ bundles on X. Recall that on $U^*_X(k)$ there is a canonical non-zero theta (Cartier) divisor Θ_k whose underlying set is $\{F \in U^*_X(k), h^0(X, F) \neq 0\}$. Put $\mathcal{M} = \mathcal{O}(\Theta_k)$. It is known that $h^0(U^*_X(k), \mathcal{M}) = 1$ ([BNR]).

Consider the natural map $\pi : SU_X(r) \times U^*_X(k) \to U^*_X(kr)$ given by tensor product. From the theorem of the square, it follows that $\pi^*\mathcal{M}$ is isomorphic to $\mathcal{L}^k \boxtimes \mathcal{M}^r$. The canonical element $\Theta_{kr} \in H^0(U^*_X(kr), \mathcal{M})$ and the Kunneth theorem gives a map well defined up to scalars:

$$H^0(SU_X(r), \mathcal{L}^k)^* \stackrel{SD}{\to} H^0(U^*_X(k), \mathcal{M}^r). \tag{1.1}$$

Let $\mathcal{X} \to S$ be a relative (smooth curve) curve with S affine. Let $X_s = \mathcal{X}_s$ for $s \in S$. We can think of X_s as a family of smooth projective curves. For convenience let $\bar{J}(X_s) = \mathrm{Jac}^{g-1}(X_s)(= U^*_{X_s}(1))$ which parameterizes line bundles of degree $g-1$ on X_s.

Assume for simplicity that the relative moduli schemes over S (see Section 4.2) carry line bundles which restrict fiberwise (upto isomorphism) to the line bundles described above (this can always be achieved locally in S by passing to open covers in the étale topology).

- The spaces $H^0(SU_{X_s}(r), \mathcal{L}^k)$ and $H^0(U^*_{X_s}(k), \mathcal{M}^r)$ organize into vector bundles $\mathcal{V}$ and $\mathcal{W}$ over S with projectively flat connections. The Hitchin/Wess-Zumino-Witten(WZW) theory gives a connection on $\mathcal{V}$, and we will define the connection on $\mathcal{W}$ by using the Galois cover $SU_{X_s}(k) \times \bar{J}(X_s) \to U^*_{X_s}(k)$.
- The map SD globalizes as well (well defined up to multiplication by $\mathcal{O}^*_S$).

The following is the main theorem of this paper:

Theorem 1.1. *The map $SD : \mathcal{V}^* \to \mathcal{W}$ is a projectively flat map of vector bundles on S.*

Analogues of the above flatness assertion are implicit in the physics papers on strange duality (e.g. the reference to the braid group in the paper [NS] where duality statements for $\mathbb{P}^1$ with insertions, are discussed). I learned from M.S. Narasimhan that the question of flatness of SD

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in the form stated above has been around for a while. It also appears in Laszlo's paper [L1], as a question suggested by Beauville.

The map SD is known to be an isomorphism. This was proved by the author [B2] for a generic curve by finding an enumerative problem with the same number of solutions as the dimension of the vector spaces that appear in SD, and then studying the implications of transversality in the enumerative problem. Subsequently, and building on ideas and strategies (see the review article of Popa [P]) from [B2], Marian and Oprea [MO] proved that SD is an isomorphism for all curves.

The flatness statement implies that the projective monodromy groups, over the moduli-stack of genus g curves coincide. It also gives an new proof of the strange duality for all curves, from the case of generic curves, see Lemma A.1. The relation between the enumerative geometry in [B2], [MO] and the projective connections remains somewhat of a mystery.

1.1. **Formulations of the main statements.** There are (at least) two equivalent ways of getting a projective connection on $H^0(SU_{X_s}(r),\mathcal{L}^k)$ (i.e. the sheaf on S with these fibers). The first one is due to Hitchin [H]. Given the identification of conformal blocks with non-abelian theta functions [V, BL, F, KNR] (which we shall refer to as the Verlinde isomorphism) we have a second way due to Tsuchiya-Ueno-Yamada, which a priori works over the moduli of pointed curves [TUY] (but in fact descends to the moduli stack of curves). This second connection is called the WZW connection. Laszlo [L2] showed that these projective connections are the same. But to impose a projective connection on $H^0(U^*_{X_s}(k),\mathcal{M}^r)$ we cannot use either of these approaches directly. We will define the projective connection on $H^0(U^*_{X_s}(k),\mathcal{M}^r)$ by using the Galois cover $SU_{X_s}(k)\times \bar{J}(X_s)\to U^*_{X_s}(k)$. Therefore we need to replace $U^*_{X_s}(k)$ by $SU_{X_s}(k)\times\bar{J}(X_s)$ (and keep track of the action of the covering group which is the group of k-torsion points in the Jacobian of X_s).

For ease of notation let $X=X_s$ which we will think of as a moving curve parameterized by $s\in S$. We begin by analyzing the objects using the diagram (1.2) (see the Appendix for the definition and properties of projective connections).

$$\begin{tikzcd}[column sep=small] & SU_X(r)\times SU_X(k)\times \bar{J}(X) \arrow[dl] \arrow[dr] & \\ SU_X(kr)\times\bar{J}(X) \arrow[dr] & & SU_X(r)\times U^*_X(k) \arrow[dl] \\ & U^*_X(kr) & \end{tikzcd} \tag{1.2}$$

(A) View Θ_{kr} as a giving a natural element (defined upto scalars)

$$\theta(r,k)\in H^0(SU_X(r),\mathcal{L}^k)\otimes H^0(SU_X(k),\mathcal{L}^r)\otimes H^0(\bar{J}(X),\mathcal{M}^{kr}) \tag{1.3}$$

induced from the natural map $SU_X(r)\times SU_X(k)\times\bar{J}(X)\to U^*_X(kr)$ which factors through $SU_X(r)\times U^*_X(k)$.

(B) All three vector spaces in (1.3) have projective connections (as X varies). The first two by Hitchin/WZW and the third from the theory of Heisenberg groups.

(C) The element $\theta(r,k)$ is the image of the element

$$\theta(kr,1) \in H^0(SU_X(kr),\mathcal{L}) \otimes H^0(\bar{J}(X),\mathcal{M}^{kr})$$

under the map

$$H^0(SU_X(kr),\mathcal{L}) \to H^0(SU_X(r),\mathcal{L}^k) \otimes H^0(SU_X(k),\mathcal{L}^r) \tag{1.4}$$

(tensored with $H^0(\bar{J}(X),\mathcal{M}^{kr})$).

We will prove the following two propositions.

Proposition 1.2. *The element $\theta(m,1) \in H^0(SU_X(m),\mathcal{L}) \otimes H^0(\bar{J}(X),\mathcal{M}^m)$ is projectively flat for any positive integer m (as X varies in a family, see Section 1.3).*

We will apply Proposition 1.2 with $m = rk$.

Proposition 1.3. *The map* (1.4)*: $H^0(SU_X(kr),\mathcal{L}) \to H^0(SU_X(k),\mathcal{L}^r) \otimes H^0(SU_X(r),\mathcal{L}^k)$ is projectively flat (as X varies in a family).*

Together these propositions imply that $\theta(r,k)$ is projectively flat (as X varies in a family). This will give Theorem 1.1 (see Section 4.4).

We can conclude that SD is an isomorphism for all curves, assuming it for generic curves, merely from the projective flatness of $\theta(r,k)$ as follows: It is enough to show that

$$H^0(SU_X(r),\mathcal{L}^k)^* \to \left(H^0(SU_X(k),\mathcal{L}^r) \otimes H^0(\bar{J}(X),\mathcal{M}^{kr})\right) \tag{1.5}$$

is injective (because we know that the image lands inside $H^0(U_X^*(k),\mathcal{M}^r)$). But (1.5) is a projectively flat map (since $\theta(r,k)$ is projectively flat and Proposition A.2), and such maps have constant rank, see Lemma A.1.

1.2. **Proofs of the propositions.** A genus 0 (with insertions, i.e. parabolic) analogue of Proposition 1.3 for conformal blocks is noted with proof in Nakanishi-Tsuchiya [NT]. Given the Verlinde isomorphism, the proof in [NT] generalizes in a straightforward manner to give Proposition 1.3. One needs to check that the Verlinde isomorphism is suitably functorial for maps of groups (this was known). The proof of Proposition 1.3 uses the fact that the embedding of Lie algebras

$$\mathrm{sl}(r) \oplus \mathrm{sl}(k) \subseteq \mathrm{sl}(rk)$$

is a conformal embedding at level 1 for $\mathrm{sl}(rk)$ (see Section 5 for more details). Indeed there is a generalization of Proposition 1.3 valid for all conformal embeddings, see Proposition 5.8 (also see [NT]). The paper [KM] is a good reference for the theory of conformal embeddings.

Proposition 1.2 is not new, although we could not find an adequate reference. It was explained to us by M. Popa that the Heisenberg group which acts irreducibly on $H^0(\bar{J}(X),\mathcal{M}^m)$ also acts on $H^0(SU_X(m),\mathcal{L})$ so that $\theta(m,1)$ induces an isomorphism $H^0(SU_X(m),\mathcal{L})^* \to H^0(\bar{J}(X),\mathcal{M}^m)$ of representations of the Heisenberg group (see [BNR] where the idea of applying the Heisenberg group already appears). Together with the arguments of Mumford [M] and Welters [W], a proof of Proposition 1.2 is easily obtained.

It would be very interesting to obtain an algebro-geometric proof of Proposition 1.3 using only Hitchin's definition of the projective connection [H] . Note that the map

$$H^0(SU_X(kr),\mathcal{L}^m) \to H^0(SU_X(r),\mathcal{L}^{mk}) \otimes H^0(SU_X(k),\mathcal{L}^{mr}),$$

is not claimed to be projectively flat (in fact very likely false) for $m > 1$. This is probably related to the discussion of compatibility of heat operators in Section 2.3.10 of [GJ].

See e.g. [SW], for a list of possible conformal embeddings (see Remark 5.12). Is there interesting enumerative geometry associated to these? According to this list (see Section 6) it is likely that the symplectic strange duality considered in [Be] is again projectively flat (also see [NT]).

1.3. **Notation and assumptions.** For technical reasons, the connection on $H^0(SU_X(r), \mathcal{L}^k)$ for every r and k (as X varies in a family) will be taken to be the WZW connection (which is a priori defined on the moduli of pointed curves, but descends to the moduli of curves). Laszlo [L2] has shown that the WZW connection is the same as Hitchin's connection if either $g > 2$ or $g = 2$ and $r \neq 2$ (in fact Hitchin's connection requires these assumptions). Our proof of Proposition 1.2 needs Laszlo's theorem and hence we need either $g > 2$ or $m > 2$ in that proposition. But in the proof of the projective flatness of $\theta(r, k)$, Proposition 1.2 is invoked for $m = rk$. Therefore, the morphism (1.5) is flat unless $g = 1$ or $g = r = 2$ and $k = 1$ (in these cases we hope that it is again flat). Perhaps, using the results of [GJ], one could show that Proposition 1.2 holds in the case $g = 2$ and $m = 2$, and that the morphism (1.5) is flat for $g > 1$.

We will permit ourselves to (sometimes) abuse notation in statements of projective flatness. For example, in Proposition 1.2 what we have in mind is the following: Start with any family of (smooth connected projective) curves $\mathcal{X} \to S$. Replacing S by an open cover in the étale topology, the spaces $H^0(SU_{X_s}(m), \mathcal{L}) \otimes H^0(\bar{J}(X_s), \mathcal{M}^m)$ form the fibers of a vector bundle $\mathcal{S}$ on S (a tensor product of suitable pushforward of line bundles from relative moduli schemes). There is a natural section θ of $\mathcal{S}$ (which is well defined locally on S up to scalars in $\mathcal{O}_S^*$). Proposition 1.2 asserts that θ is a projectively flat section of $\mathcal{S}$.

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2. Heisenberg groups

Let X be a smooth projective and connected curve of genus g. Let $J(X) = \mathrm{Jac}^0(X)$, and $\bar{J}(X) = \mathrm{Jac}^{g-1}(X)$ as in the introduction. For $a \in J(X)$ we have a natural translation map $T_a : \bar{J}(X) \to \bar{J}(X)$. The finite Heisenberg group $\mathcal{G}_X(m)$ is defined to be the collection of pairs (a, ψ) where $a \in J(X)$ and ψ an isomorphism $\mathcal{M}^m \to T_a^*\mathcal{M}^m$ ($\mathcal{M}$ is the line bundle on $\bar{J}(X)$ defined in the introduction). The canonical reference for Heisenberg groups is the series of papers of Mumford [M].

Clearly, $\mathcal{G}(m)$ is a central extension:

$$1 \to \mathbb{C}^* \to \mathcal{G}_X(m) \to \mathcal{H}_X(m) \to 0 \tag{2.1}$$

where $\mathcal{H}_X(m) \subseteq J(X)$ as a subgroup. Since $\mathcal{M}$ is a principal polarization, $\mathcal{H}_m(m)$ is the group of m-torsion points in $J(X)$.

Now, consider the map $\pi : SU_X(m) \times \bar{J}(X) \to U_X^*(m)$ and fix an isomorphism $\pi^*\mathcal{M} \xrightarrow{\delta} \mathcal{L} \boxtimes \mathcal{M}^m$. Define an action of $\mathcal{G}_X(m)$ on $(SU_X(m), \mathcal{L})$ and $(U_X^*(m), \mathcal{M})$ as follows: Let $(L_0, \psi) \in \mathcal{G}_X(m)$

(1) The action on $(U_X^*(m), \mathcal{M})$ is trivial.
(2) The action on $SU_X(m)$ is by tensoring with L_0^{-1}. The action on $\mathcal{L}$ is obtained as follows: At $E \in SU_X(m)$ and $L \in \bar{J}(X)$, we have a map
$$\mathcal{L}_E \otimes \mathcal{M}_L^m \to \mathcal{L}_{E\otimes L_0^{-1}} \otimes \mathcal{M}_{L\otimes L_0}^m,$$
because both sides are identified with the fiber of $\mathcal{M}$ at $E \otimes L \in U_X^*(m)$. The isomorphism ψ therefore gives us an isomorphism
$$\mathcal{L}_E \to \mathcal{L}_{E\otimes L_0^{-1}}$$
which may a priori depend upon L, but does not, because otherwise (fixing E) we would get a non-constant function on $\bar{J}(X)$ with values in a one dimensional vector space.

Notice that changing δ (by scale) does not change the action of $\mathcal{G}_X(m)$ on $(SU_X(m), \mathcal{L})$. The action of $\mathcal{G}_X(m)$ clearly extends to an action on the pairs $(SU_X(m), \mathcal{L}^k)$ and $(\bar{J}(X), \mathcal{M}^{km})$, and a trivial action on the pair $(U_X^*(m), \mathcal{M}^k)$.

Lemma 2.1. (1) [BNR] *The vector spaces* $H^0(\bar{J}(X), \mathcal{M}^m)$ *and* $H^0(SU_X(m), \mathcal{L})$ *are dual representations of the Heisenberg group* $\mathcal{G}_X(m)$*, and are both irreducible.*
(2) $\big(H^0(SU_X(m), \mathcal{L}^r) \otimes H^0(\bar{J}(X), \mathcal{M}^{mr})\big)^{\mathcal{G}_X(m)} \xrightarrow{\sim} H^0(U_X^*(m), \mathcal{M}^r)$ *with the isomorphism depending on the choice of* δ*, in a one dimensional space.*
(3) $\mathcal{G}_X(m) \subseteq \mathcal{G}(mr)$ *with compatible action on* $(\bar{J}(X), \mathcal{M}^{mr})$.

Proof. We know from [BNR] that the ranks of $H^0(\bar{J}(X), \mathcal{M}^m)$ and $H^0(SU_X(m), \mathcal{L})$ agree. By Mumford's theory, $H^0(\bar{J}(X), \mathcal{M}^m)$ is an irreducible representation of $\mathcal{G}_X(m)$. Therefore any non-zero element in
$$\big(H^0(SU_X(m), \mathcal{L}) \otimes H^0(\bar{J}(X), \mathcal{M}^m)\big)^{\mathcal{G}_X(m)} = H^0(U_X^*(m), \mathcal{M})$$
gives a non zero $\mathcal{G}_X(m)$-equivariant map from $H^0(SU_X(m), \mathcal{L})$ to the dual of $H^0(\bar{J}(X), \mathcal{M}^m)$ which is necessarily an isomorphism of representations of $\mathcal{G}_X(m)$. This proves (1). The assertions (2) and (3) are clear. □

3. Welters's deformation theory

Let us recall some aspects of Welters's deformation theory of pairs (see [W], and [L2], Section 6). Let X be a smooth variety and L a line bundle on X.

By the classical Kodaira-Spencer theory, the deformations of X over $\operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2)$ are classified by elements in $H^1(X, T_X)$. The deformation of pairs (X, L) over $\operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2)$ are classified by elements in $H^1(X, \mathcal{D}^1(L))$ (where $\mathcal{D}^i(L)$ is the sheaf of differential operators of order $\leq i$ on L). The natural ("symbol") map $D^1(L) \to T_X$ on H^1 gives the map from deformations of pairs (X, L) to deformations of X.

Let s be a global section of L over X. Let $d^i s$ denote the complex $\mathcal{D}^i(L) \xrightarrow{s} L$ (with $\mathcal{D}^i(L)$ in degree 0 and L in degree 1). According to Welters, the deformations of the triple (X, L, s) are classified by elements of the hypercohomology group $H^1(d^1 s)$.

Now let $A \in H^0(S^2(T_X))$. Welters considers the exact sequence of complexes obtained from the symbol map
$$0 \to d^1 s \to d^2 s \to S^2(T_X) \to 0$$
to produce an element in $H^1(X, d^1 s)$. Therefore elements of $H^0(S^2 T_X)$ deform all triples (X, L, s).

3.1. **Compatibility under automorphisms.** Let X_ϵ be a smooth over $D_\epsilon = \operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2)$, and L_ϵ a line bundle over it. Assume that $H^0(X_\epsilon, \mathcal{O}_{X_\epsilon}) = \mathcal{O}_{D_\epsilon}$.

Let A be a global section of S^2T_X (where X is the fiber over 0). The deformation (X_ϵ, L_ϵ) produces a class in $H^1(X, \mathcal{D}^1(L))$. The element A also produces a class in the same group $H^1(X, \mathcal{D}^1(L))$. Assume that these two classes agree.

Now suppose in addition that we have an automorphism ψ_ϵ of (X_ϵ, L_ϵ) over D_ϵ and a section s of L over X. By Welters's theory, A induces a deformation of the section s as well. That is, A induces a global section s_ϵ of L_ϵ which restricts to s. The resulting s_ϵ is unique up to automorphisms of L_ϵ which are trivial over the central fiber ($= 1 + \epsilon\mathbb{C}$ in the case at hand). Then

Lemma 3.1. *Let $\psi = \psi_0$ and suppose that $\psi_* A = A$. Then,* $\psi_\epsilon s_\epsilon = (\psi s)_\epsilon \pmod{1 + \epsilon\mathbb{C}}$

Proof. Consult (all) diagrams on page 16 of [W]. □

4. Hitchin's connection

Consider a $E \in SU_X^0(m)$ (the set of regularly stable points). The tangent space to $SU_X^0(m)$ at E is $H^1(X, \operatorname{End}_0(E))$, where $\operatorname{End}_0(E)$ is the sheaf of trace 0 endomorphisms of E. The cotangent space is therefore, by Serre duality, equal to $H^0(X, \operatorname{End}_0(E) \otimes \Omega_X^1)$. An infinitesimal deformation of a curve is parameterized by $t \in H^1(X, T_X)$. Give such a t, one obtains a map

$$H^0(X, \operatorname{End}_0(E) \otimes \Omega^1) \otimes H^0(X, \operatorname{End}_0(E) \otimes \Omega^1) \to \mathbb{C}$$

by taking the killing form of the pair of endomorphisms and contracting the product of the two 1 forms with t (at the level of Cech cochains), and finally taking the trace (which is a map $H^1(X, \Omega_X^1) \to \mathbb{C}$). Therefore we obtain an element $\tau(t) \in S^2(T_{SU_X^0(m)})$. The following is immediate:

Lemma 4.1. *Let L_0 be an m-torsion line bundle on X. Then the automorphism of $SU_X^0(m)$ obtained as tensoring with L_0 preserves the quadratic vector field $\tau(t)$.*

4.1. **Properties of Hitchin's connection.** Let $\mathcal{X} \to S$ be a family of curves, as before $X = \mathcal{X}_s$ with $s \in S$, and $\tilde{t} \in TS_s$. We have a family of moduli-spaces $(SU_{X_s}^0(m), \mathcal{L})$. Base change this to the corresponding family over $S = \operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2)$.

The element $\tilde{t}$ produces an element $t \in H^1(X, T_X)$, which through $-\frac{\tau(t)}{2m+2k}$ brings about a deformation in the pair $(SU_X^0(m), \mathcal{L}^k)$. This deformation agrees with the geometric deformation of the previous paragraph (see [L2]). The deformation in triples $(SU_X^0(m), \mathcal{L}^k, s)$ produced by $-\frac{\tau(t)}{2m+2k}$ is the Hitchin connection (the projective ambiguity arises out of automorphisms of $\mathcal{L}_\epsilon$ that are trivial over the central fiber): the (first-order) parallel transport of s along $\tilde{t}$ is the deformed section s_ϵ.

Now note that by codimension considerations (see [L2]), $H^0(SU_{X_s}^0(m), \mathcal{L}^k) = H^0(SU_{X_s}(k), \mathcal{L}^k)$.

4.2. **Heisenberg group schemes.** Let $\mathcal{X} \to S$ be a smooth curve. For simplicity (by passing to étale covers) assume that the sheaf of m torsion points in the Jacobian of the curves X_s is trivial on S.

Assume that we have relative pairs $(\underline{\bar{J}}, \underline{\mathcal{M}})$, $(\underline{U}^*(m), \underline{\mathcal{M}})$ and $(\underline{SU}(m), \underline{\mathcal{L}})$ of (schemes,line bundles) over S with fibers $(\bar{J}(X_s), \mathcal{M})$, $(U_{X_s}^*(m), \mathcal{M})$ and $(SU_{X_s}(m), \mathcal{L})$ over $s \in S$, such that

the line bundles $\mathcal{M}$ and $\mathcal{L}$ are isomorphic to the line bundles defined in the introduction. One can always replace S by a cover in the étale topology to ensure this. The line bundles on the relative moduli schemes are unique up to tensoring with line bundles from S.

We can form a group scheme $\mathcal{G}(m)$ over S whose fiber over $s \in S$ is the group scheme $\mathcal{G}_{X_s}(m)$ from Section 2 (see [W]). All constructions in Section 2 carry over to this situation. In particular there is an action of $\mathcal{G}(m)$ on $p_*\underline{\mathcal{L}}^k$ and $q_*\underline{\mathcal{M}}^m$ (for any k) where p and q denote the maps $\underline{SU}(m) \to S$ and $\underline{\bar{J}} \to S$ respectively.

Fix $b \in S$. Replace S by a connected étale neighborhood U of b such that there is an isomorphism of group schemes $\lambda : \mathcal{G}(m) \to \mathcal{G}_{X_b}(m) \times_{\mathbb{C}} U$ inducing the identity over b and commuting with the projection to the sheaf of m-torsion points in the Jacobian. Using the exact sequence (2.1), note that λ is unique. We will keep this notation and assumption fixed for the rest of Section 4. Therefore elements of the fixed group $\mathcal{G}_{X_b}(m)$ act on the sheaves $p_*\underline{\mathcal{L}}^k$ and $q_*\underline{\mathcal{M}}^m$ on S.

From Lemmas 3.1 and 4.1, we conclude:

Corollary 4.2. *The action of the group $\mathcal{G}_{X_b}(m)$ on $p_*\underline{\mathcal{L}}^k$ preserves Hitchin's connection ∇: That is, for every $h \in \mathcal{G}_{X_b}(m)$, there exists a one-form ω_h such that*

$$h\nabla(v) - \nabla(hv) = \omega_h hv \tag{4.1}$$

for all sections v of $p_*\underline{\mathcal{L}}^k$.

Proof. Indeed by Lemmas 3.1, 4.1 and A.5 applied to ∇ and $h^{-1}\nabla h$, there exists an one-form ω_h on S such that equation (4.1) holds. □

4.3. **Proof of Proposition 1.2.** Let us recall how one obtains a (projective) connection on $q_*\underline{\mathcal{M}}^m$ through the theory of Heisenberg groups (for more details see [W]). The representation $H^0(\bar{J}(X_s), \mathcal{M}^m)$ is the *unique* irreducible representation of $\mathcal{G}_{X_s}(m)$ on which the central $\mathbb{C}^*$ acts by the basic character ($z \in \mathbb{C}^*$ acts by multiplication by z). Since the Heisenberg group scheme $\mathcal{G}(m)$ is trivialized over the base S, we can identify any $H^0(\bar{J}(X_s), \mathcal{M}^m)$ (the fiber of $q_*\underline{\mathcal{M}}^m$ at s) with this basic representation (up to scalars). The parallel transport is immediate and hence the (projective) connection. It follows from [W] that $\mathcal{G}_{X_b}(m)$ acts in a projectively flat manner on $q_*\underline{\mathcal{M}}^m$.

It now follows from Propositions 4.2 and A.3 that the subsheaf $(p_*\underline{\mathcal{L}} \otimes q_*\underline{\mathcal{M}}^m)^{\mathcal{G}_{X_b}(m)}$ is preserved by the product connection on $p_*\underline{\mathcal{L}} \otimes q_*\underline{\mathcal{M}}^m$ (Hitchin⊗1 + 1⊗"Heisenberg"). It is clear that $(p_*\underline{\mathcal{L}} \otimes q_*\underline{\mathcal{M}}^m)^{\mathcal{G}_{X_b}(m)}$ can be calculated fiberwise (see Remark A.4), and we find that it is a one dimensional $\mathcal{O}_S$ module. Any local generator of it gives a projectively flat section. This gives Proposition 1.2.

4.4. **Proof of Theorem 1.1 assuming Proposition 1.3.** We can view $\theta(r,k)$ as a projectively flat element of the sheaf on S with fibers

$$H^0(SU_{X_s}(r), \mathcal{L}^k) \otimes \big(H^0(SU_{X_s}(k), \mathcal{L}^r) \otimes H^0(\bar{J}(X_s), \mathcal{M}^{kr})\big)^{\mathcal{G}_{X_s}(k)}$$

The group scheme over S with fiber $\mathcal{G}_{X_s}(k)$ over s acts in a projectively flat manner on the sheaves on S with fibers $H^0(SU_{X_s}(k), \mathcal{L}^r)$ and $H^0(\bar{J}(X_s), \mathcal{M}^{kr})$ (see Section 4.3), and the space

$$\big(H^0(SU_{X_s}(k), \mathcal{L}^r) \otimes H^0(\bar{J}(X_s), \mathcal{M}^{kr})\big)^{\mathcal{G}_{X_s}(k)}$$

of invariants is canonically $H^0(U^*_{X_s}(k),\mathcal{M}^r)$. This will impose a projective connection on $H^0(U^*_{X_s}(k),\mathcal{M}^r)$ such that SD is projectively flat, see Lemmas A.3 and A.2.

5. Conformal blocks and the WZW connection

5.1. **Conformal blocks.** Let us first begin with the case of a fixed curve X, a semisimple simply connected complex algebraic group G, and state the Verlinde isomorphism comparing conformal blocks and non-abelian G-theta functions ([BL, F, KNR]). We find the stack theoretic treatment given in [BL, LS, BLS] suitable for our purposes.

Fix $p\in X$ and a local parameter z at p. Let $K=\mathbb{C}((z))$ (formal meromorphic laurent series) and $O=\mathbb{C}[[z]]$ and $A_X=\mathcal{O}(X-p)$. Let $LG=G(K), L^+G=G(O), L_X(G)=G(A_X)$. Suppose further that $G=\prod_{i=1}^k G_i$.

Let $\hat{\mathfrak{g}}$ denote the Kac-Moody Lie algebra of G which equals $\oplus_{i=1}^k\hat{\mathfrak{g}}_i$ where each $\hat{\mathfrak{g}}_i$ is a central extension of $\mathfrak{g}_i\otimes K$ by $\mathbb{C}c_i$. There is an embedding of Lie algebras $\mathfrak{g}\otimes A_X\to\hat{\mathfrak{g}}$. Given $\ell=(\ell_1,\dots,\ell_k)\in\mathbb{Z}^k_{\geq 0}$, denote by V_ℓ the basic irreducible representation of $\hat{\mathfrak{g}}$ at level ℓ. It is known that V_ℓ is a tensor product of basic representations of level ℓ_i of $\hat{\mathfrak{g}}_i$.

Let $\mathcal{M}_G=\mathcal{M}_G(X)$ denote the moduli-stack of G-bundles on X and $\mathcal{Q}_G=LG/L^+G$ the infinite Grassmannian (an ind-scheme).

The uniformization theorem of Beauville and Laszlo gives a canonical isomorphism of stacks:

$$L_XG\backslash\mathcal{Q}_G\to\mathcal{M}_G(X)$$

The Picard group of $\mathcal{M}_G$ equals $\oplus_{i=1}^k\mathbb{Z}$. Given $\ell=(\ell_1,\dots,\ell_k)\in\mathbb{Z}^k_{\geq 0}$ let $\mathcal{L}(\ell)$ denote the corresponding line bundle on $\mathcal{M}_G$. The space of sections of the pull back of the line bundle $\mathcal{L}(l)$ to $\mathcal{Q}_G$ equals the dual of V^*_ℓ. Upon identification of the pull back of $\mathcal{L}(l)$ to $\mathcal{Q}_G$, this is a consequence of a theorem of Kumar [K] and Mathieu [Ma].

For $\ell\in\mathbb{Z}^k_{\geq 0}$, the Verlinde isomorphism gives is a canonical isomorphism (up to scalars)

$$H^0(\mathcal{M}_G,\mathcal{L}(\ell))\xrightarrow{\sim}(V^*_\ell)^{\mathfrak{g}\otimes A_X}=\{\phi\in V^*_\ell\mid\phi(Mv)=0,\forall\ M\in\mathfrak{g}\otimes A_X, v\in V_\ell\}\tag{5.1}$$

The vector space on the right hand side of (5.1) is the called the space of conformal blocks, associated to data (X,p,z). We will call $H^0(\mathcal{M}_G,\mathcal{L}(\ell))$ the space of non-abelian G-theta functions on X.

Now assume that $G\to H$ is a morphism of algebraic groups where H is *simple* (for simplicity!) simply connected, complex algebraic group. In this situation, there is a Dynkin index $d=(d_1,\dots,d_k)\in\mathbb{Z}^k_{>0}$ so that

(1) The generating line bundle in $\mathcal{M}_H$ pulls back to the line bundle with indices $(d_1,\dots,d_k)$ on $\mathcal{M}_G$.
(2) There is an induced map $\hat{\mathfrak{g}}\to\hat{\mathfrak{h}}$ which maps c_i to d_i times the generating central element in $\hat{\mathfrak{h}}$ (here c_i is the generating central element of $\hat{\mathfrak{g}}_i$).

Now given a basic level $p>0$ representation of $\hat{\mathfrak{h}}$ with highest weight vector v, there is a unique $\hat{\mathfrak{g}}$ representation with highest weight vector v inside V_p which is canonically (up to scalars) isomorphic to the representation V_ℓ of $\hat{\mathfrak{g}}$ of level $\ell=(pd_1,\dots,pd_k)$.

Remark 5.1. *Note that we do not assume $G\to H$ to be compatible with the Borel subgroups, because we are in the case where the corresponding representations of the ordinary Lie algebras are trivial.*

The following proposition studies the functoriality of the Verlinde isomorphism (5.1).

Proposition 5.2. *Let $\mathcal{L}$ be the generator of the Picard group of $\mathcal{M}_H$. The following diagram commutes (up to scalars), where the vertical map on the right hand side is induced by the inclusion $V_\ell \subseteq V_p$ described above:*

$$\begin{tikzcd} H^0(\mathcal{M}_H, \mathcal{L}^p) \arrow[r] \arrow[d] & (V_p^*)^{\mathfrak{h}\otimes A_X} \arrow[d] \\ H^0(\mathcal{M}_G, \mathcal{L}(\ell)) \arrow[r] & (V_\ell^*)^{\mathfrak{g}\otimes A_X} \end{tikzcd} \tag{5.2}$$

Proof. Consider the (2-commutative in the sense of stacks) diagram

$$\begin{tikzcd} \mathcal{Q}_G \arrow[r, "\pi"] \arrow[d] & \mathcal{M}_G \arrow[d] \\ \mathcal{Q}_H \arrow[r, "\pi"] & \mathcal{M}_H \end{tikzcd}$$

□

Therefore we have to show that the map $H^0(\mathcal{Q}_H, \pi^*\mathcal{L}^p) \to H^0(\mathcal{Q}_G, \pi^*\mathcal{L}(\ell))$ is projectively identified with $V_p^* \to V_\ell^*$. But this follows from the following commutative diagram of ind-schemes

$$\begin{tikzcd} \mathcal{Q}_G \arrow[r, "\gamma_\ell"] \arrow[d] & \mathbb{P}(V_\ell) \arrow[d] \\ \mathcal{Q}_H \arrow[r, "\gamma_p"] & \mathbb{P}(V_p) \end{tikzcd}$$

and the identifications $\gamma_\ell^*\mathcal{O}(1) = \mathcal{L}(\ell)$ (similarly for γ_p) and $H^0(\mathbb{P}(V_\ell), \mathcal{O}(1)) = V_\ell^*$ (similarly for $\mathbb{P}(V_p)$). Here γ_ℓ is the map that takes $g \in LG$ to $[gv]$ and γ_p takes $h \in LH$ to $[hv]$ (note that LG acts projectively on V_ℓ and LH on V_p).

5.2. **Representations of Virasoro algebras.** Recall that V_ℓ is an irreducible representation of the Kac-Moody Lie algebra $\hat{\mathfrak{g}}$. We will now describe the action of the Lie algebra of continuous derivations of $\mathbb{C}((z))$ (called the Virasoro algebra) on V_ℓ (see Remark 5.4). We will define such an action for any reasonable representation of $\hat{\mathfrak{g}}$ following [KM].

5.2.1. *Virasoro algebras.* Let $S_n = -z^{n+1}\frac{d}{dz}$ for $n \in \mathbb{Z}$, as vector fields. It is easy to see that $[S_j, S_k] = (j-k)S_{j+k}$. The Virasoro algebra Vir is a complex Lie algebra with basis $\{\tilde{c}, d_j, j \in \mathbb{Z}\}$ and the commutation relations

$$[d_j, d_k] = (j-k)d_{j+k} + \frac{1}{12}(j^3 - j)\delta_{j,-k}\tilde{c},\ [d_j, \tilde{c}] = 0.$$

A Lie algebra representation V of Vir is said to have central charge m if $\tilde{c}$ acts by multiplication by m on V. We will represent such a representation by (A_n, m) where A_n is the endomorphism of V given by the action of d_n, and m is the central charge.

5.2.2. Vir-*representations from the Segal-Sugawara construction.* For $x \in \mathfrak{g}$ and $d \in \mathbb{Z}$, let $x(d) = z^d \otimes x \in \hat{\mathfrak{g}}$. Now let V be any (not necessarily irreducible) representation on $\hat{\mathfrak{g}}$ which satisfies

(C1) For all $v \in V$ and $x \in \mathfrak{g}$, $x(d)v = 0$ for d sufficiently large.
(C2) The central elements c_i in $\mathfrak{g}$ act as positive scalars m_i on V.

Case $\mathfrak{g}$ simple: We will first define the action of Vir on V assuming first that $\mathfrak{g}$ is simple. Therefore assume that the central element c in $\hat{\mathfrak{g}}$ acts on V by a positive scalar m.

Normalize the Killing form by requiring that $(\theta, \theta) = 2$. Let g be the dual Coxeter number of the simple lie algebra $\mathfrak{g}$. Choose dual basis u_i and u^i of $\mathfrak{g}$ and put (see [KM], page 43)

$$L_n^{\hat{\mathfrak{g}}} = \frac{1}{2(m+g)} \sum_{j \in \mathbb{Z}} \sum_i : u_i(-j) u^i(j+n) :$$

Here $: u(s)v(r) :$ stands for $u(s)v(r)$ if $s \leq r$ and $v(r)u(s)$ if $s > r$. It is known that defining the action of $\tilde{c}$ as multiplication by $z_m = \frac{(\dim \mathfrak{g})m}{g+m}$, and the action of d_n by L_n gives an action of Vir on V of central charge z_m.

Case $\mathfrak{g}$ arbitrary: We set $L_n^{\hat{\mathfrak{g}}} = \sum_{i=1}^k L_n^{\hat{\mathfrak{g}}_i}$. We obtain a representation on Vir on V of central charge

$$\sum_{i=1}^k \frac{(\dim \mathfrak{g}_i) m_i}{g_i + m_i}$$

where g_i is the dual Coxeter number of $\mathfrak{g}_i$.

Definition 5.3. *For $t = \sum_{n \geq -N} t_n S_n \in \mathbb{C}((z))\frac{d}{dz}$, define the following operator on V:*

$$T^{\hat{\mathfrak{g}}}(t) = \sum_{n \geq -N} t_n L_n^{\mathfrak{g}}$$

(this is a finite sum).

Remark 5.4. *It is known that for $x \in \hat{\mathfrak{g}}$, $[x, T^{\hat{\mathfrak{g}}}(t)] = t.x$ as operators on V. Therefore the (continuous) derivations t of $\mathbb{C}((t))$ lift to operators $T^{\hat{\mathfrak{g}}}(t)$ on V, compatible with the action of t on $\hat{\mathfrak{g}}$.*

5.2.3. *Coset Virasoro representations.* Let $\mathfrak{g} \subset \mathfrak{h}$ be an embedding of semisimple Lie algebras with $\mathfrak{h}$ simple. There is an induced homomorphism $\hat{\mathfrak{g}} \to \hat{\mathfrak{h}}$. Assume that $\mathfrak{g} = \sum_i \mathfrak{g}_i$ and that c_i map to cd_i. Let V be a representation of $\hat{\mathfrak{h}}$ that satisfies (C1) and (C2) such that the center of $\hat{\mathfrak{h}}$ acts by multiplication by p. Then, considered as a representation $\hat{\mathfrak{g}}$, V satisfies (C1) and (C2) as well. The central element c_i in $\hat{\mathfrak{g}}$ acts by multiplication by pd_i.

Therefore we have two representations of Vir on V represented by $(L_n^{\hat{\mathfrak{g}}}, a_{\hat{\mathfrak{g}}})$ and $(L_n^{\hat{\mathfrak{h}}}, a_{\hat{\mathfrak{h}}})$. Here

$$a_{\hat{\mathfrak{g}}} = \sum_{i=1}^k \frac{(\dim \mathfrak{g}_i) p d_i}{g_i + p d_i}$$

and

$$a_{\hat{\mathfrak{h}}} = \frac{(\dim \mathfrak{h}) p}{g(\mathfrak{h}) + p}$$

where $g(\mathfrak{h})$ is the dual Coxeter number of $\mathfrak{h}$.

Now there is a remarkable "difference" representation of Vir [GKO] (also see [KM] and [K], chapter 12) on V. This representation of Vir represented by $(L_n^{\hat{\mathfrak{h}}} - L_n^{\hat{\mathfrak{g}}}, a_{\hat{\mathfrak{h}}} - a_{\hat{\mathfrak{g}}})$, is called the coset representation of Vir.

If V is a basic representation of a level $p > 0$ of $\hat{\mathfrak{h}}$, then this coset Vir-representation has been studied closely (see [KM], page 200). We need one aspect of this beautiful theory: If the central charge of the coset representation of Vir is zero, then the coset Vir representation is trivial ([K], Proposition 11.12 and [KM], Proposition 3.2 (c)). Hence

Proposition 5.5. *If V is basic representation of $\mathfrak{h}$ at a positive integer level p, and $a_{\hat{\mathfrak{g}}} = a_{\hat{\mathfrak{h}}}$, then $L_n^{\hat{\mathfrak{h}}} = L_n^{\hat{\mathfrak{g}}}$ as operators on V for all $n \in \mathbb{Z}$. Equivalently, for all $t \in \mathbb{C}((z))\frac{d}{dz}$, $T^{\hat{\mathfrak{g}}}(t) = T^{\hat{\mathfrak{h}}}(t)$ as endomorphisms of V.*

Remark 5.6. *In* [KM]*, for ease of calculation, one starts with not a basic representation of $\hat{\mathfrak{h}}$ but of the Lie algebra $\hat{\mathfrak{h}} + \mathbb{C}d$ where d brackets with $\hat{\mathfrak{h}}$ as $z\frac{d}{dz}$ and commutes with the center. It is easy to see that the relevant representation of $\hat{\mathfrak{h}}$ extends to $\hat{\mathfrak{h}} + \mathbb{C}d$. (See* [KM]*, Section 1.5 and the introduction).*

Definition 5.7. *An embedding $\mathfrak{g} \subseteq \mathfrak{h}$ of lie algebras is said to conformal at level p if $a_{\hat{\mathfrak{g}}} = a_{\hat{\mathfrak{h}}}$ for the basic representation V_p of $\hat{\mathfrak{h}}$.*

Curiously conformal embeddings (with $\mathfrak{h}$ simple and $\mathfrak{g} \subsetneq \mathfrak{h}$) always have $p = 1$. Therefore the condition on p is usually omitted. The first case when this happens, crucial for strange duality is $sl(r) \oplus sl(k) \subseteq sl(rk)$, and V a level 1 representation of $\hat{sl}(rk)$, in this case $(d_1, d_2) = (k, r)$ and the central charges are

$$a_{\hat{\mathfrak{h}}} = \frac{(rk)^2 - 1}{rk + 1}$$
$$a_{\hat{\mathfrak{g}}} = \frac{(r^2 - 1)k}{k + r} + \frac{(k^2 - 1)r}{r + k}$$

which are easily seen to be the same.

Another case which corresponds to the symplectic strange duality is $sp(2r) \oplus sp(2k) \subseteq so(4mn)$ and V a level 1 representation of $\hat{so}(4mn)$, in this case $(d_1, d_2) = (k, r)$ and the central charges are

$$a_{\hat{\mathfrak{h}}} = \frac{2rk(4rk - 1)}{4rk - 2 + 1}$$
$$a_{\hat{\mathfrak{g}}} = \frac{r(2r + 1)k}{k + r + 1} + \frac{k(2k + 1)r}{r + k + 1}.$$

which are again equal. The complete list of conformal embeddings appears in [SW].

5.3. **The WZW connection.** Let $\pi : \mathcal{X} \to S$ be a smooth relative curve over a smooth base S of arbitrary fiber genus. Suppose that we are given a section $\sigma : S \to \mathcal{X}$ of π and a formal coordinate along the fibers of π along the section σ (so that σ is identified with $z = 0$):

$$\hat{\mathcal{O}}_{\mathcal{X},\sigma} \xrightarrow{\sim} \mathcal{O}_S[[z]]$$

Let $s \in S$ and $\tau \in TS_s$. Pick a formal vector field $t \in \mathbb{C}((z))\frac{d}{dz}$ that corresponds to τ. (More precisely, we choose a local section of the map τ on page 15 in [S].)

We will describe the connections on the sheaf of dual of conformal blocks on S. This sheaf is a quotient of $V_\ell \otimes \mathcal{O}_S$, and the fiber over any $s \in S$ is the space $V_\ell/\mathfrak{g} \otimes A_{\mathcal{X}_s} V_\ell$ (note that it is a basic property that conformal blocks base change "correctly").

The WZW connection Δ on the sheaf of conformal blocks arises as follows: Let $u \in V_\ell$ and $f \in \mathcal{O}_S$. Then

$$\Delta_\tau(u \otimes f) = u \otimes \tau.f \ + \ (T^{\hat{\mathfrak{g}}}(t)u) \otimes f \quad (\text{mod } u \otimes f).$$

This operation descends to the sheaf of dual conformal blocks and hence to its dual, the sheaf of conformal blocks. We thus obtain a projective connection on the sheaf of G-nonabelian theta functions on S as well, which is independent of the choice of the section σ and the formal coordinate on the fibers along σ (e.g. as a consequence of Laszlo's comparison theorem [L2]).

Proposition 5.8. *Assume that $\mathfrak{g} \subseteq \mathfrak{h}$ is a conformal embedding at level p. Let $G \to H$ be the associated map of simply connected complex algebraic groups, and $\mathcal{X} \to S$ a smooth relative curve. Then the map $H^0(\mathcal{M}_H(X_s), \mathcal{L}^p) \to H^0(\mathcal{M}_G(X_s), \mathcal{L}(\ell))$ is projectively flat for the WZW connection.*

Proof. We can assume that we have a section of $\mathcal{X} \to S$ (by passing to a cover of S in the étale topology) and fix a formal coordinate along the section to verify the given assertion. Given the Verlinde isomorphism (5.1), it is enough to show that under the inclusion $V_\ell \subseteq V_p$, there is an equality of Sugawara operators $T^{\hat{\mathfrak{g}}} = T^{\hat{\mathfrak{h}}}$ (as operators on V_ℓ). But this is immediate from Proposition 5.5. $\square$

Remark 5.9. *An obvious extension of Proposition 5.8 holds for semisimple $\mathfrak{h}$ (where we require equality of central charges). One may be tempted to apply it to the diagonal embedding $G \subset G \times G$. But the central charges are never equal (so the multiplication map on theta functions is not claimed to be projectively flat).*

Note that if G_1 and G_2 are two groups, then there is a 1-isomorphism of stacks $\mathcal{M}_{G_1}(X) \times \mathcal{M}_{G_2}(X) \to \mathcal{M}_{G_1 \times G_2}(X)$. Therefore, Proposition 5.8 yields Proposition 1.3. (In the setting of Proposition 1.3, we need to pass from the moduli-stack to the moduli space, but this is known from [BL].)

Let us apply Proposition 5.8 to the example of symplectic strange duality. Under the map $\mathcal{M}_{\mathrm{Sp}(2m)} \times \mathcal{M}_{\mathrm{Sp}(2n)} \to \mathcal{M}_{\mathrm{Spin}(4mn)}$, the generating line bundle $\mathcal{P}$ of the stack $\mathcal{M}_{\mathrm{Spin}(4mn)}$ pulls back to $\mathcal{L}^n \boxtimes \mathcal{L}^m$, where $\mathcal{L}$ denotes the generating line bundle of the moduli stack $\mathcal{M}_{\mathrm{Sp}(2n)}$ (and of $\mathcal{M}_{\mathrm{Sp}(2n)}$).

Proposition 5.10. *The map*

$$H^0(\mathcal{M}_{\mathrm{Spin}(4mn)}(X), \mathcal{P}) \to H^0(\mathcal{M}_{\mathrm{Sp}(2m)}(X), \mathcal{L}^n) \times H^0(\mathcal{M}_{\mathrm{Sp}(2n)}(X), \mathcal{L}^m)$$

is projectively flat (as X varies in a family).

In the above proposition we may replace $H^0(\mathcal{M}_{\mathrm{Sp}(2m)}(X), \mathcal{L}^n)$ and $H^0(\mathcal{M}_{\mathrm{Sp}(2n)}(X), \mathcal{L}^m)$, by global sections over the moduli spaces (of suitable line bundles: the line bundle $\mathcal{L}$ descends to the moduli space). We cannot replace $\mathcal{M}_{\mathrm{Spin}(4mn)}(X)$ by the corresponding moduli space (but we can do so if we replace $\mathcal{M}_{\mathrm{Spin}(4mn)}(X)$ by the regularly stable part of the moduli-space).

Let us now consider an exotic example: the embedding $so_m \subseteq sl_m$ at level 1. The Dynkin index is 2 and the central charges are $\frac{(2(m^2-m)/2)}{m-2+2}$ and $\frac{m^2-1}{m+1}$ which are equal. Therefore we conclude

Proposition 5.11. *The map*

$$H^0(\mathcal{M}_{\mathrm{SL}(m)}(X), \mathcal{L}) \to H^0(\mathcal{M}_{\mathrm{Spin}(m)}(X), \mathcal{P}^2)$$

is projectively flat (as X varies in a family) where $\mathcal{L}$ and $\mathcal{P}$ are positive generators of the Picard groups of $\mathcal{M}_{\mathrm{SL}(m)}(X)$ and $\mathcal{M}_{\mathrm{Spin}(m)}(X)$ respectively.

Remark 5.12. *There is a more general definition of the notion of conformal pairs, where we not require the Lie algebras to be semisimple (but require reductiveness). However, we do not know how to make use of this more general definition, when the groups involved are not semisimple. For example, does the (conformal) embedding $gl(m) \subseteq so(2m)$ (see* [SW]*) imply that a certain map of non-abelian theta functions is projectively flat?*

6. Symplectic strange duality

Consider the moduli stack $\mathcal{M}_{\mathrm{Spin}(r)}$ of $\mathrm{Spin}(r)$-bundles on a smooth projective curve X. There is a natural map

$$p : \mathcal{M}_{\mathrm{Spin}(r)} \to \mathcal{M}_{\mathrm{SO}(r)}(0).$$

(Here $\mathcal{M}_{\mathrm{SO}(r)}(0)$ is a connected component of the moduli-stack $\mathcal{M}_{\mathrm{SO}(r)}$, see [LS, BLS])

For each theta-characteristic κ on X there is a line bundle $\mathcal{P}_k$ on $\mathcal{M}_{\mathrm{SO}(r)}(0)$ with a canonical section s_κ (see the Pfaffian construction in [LS, BLS]). The various κ give non-isomorphic line bundles on $\mathcal{M}_{\mathrm{SO}(r)}(0)$, but their pull backs to $\mathcal{M}_{\mathrm{Spin}(r)}$ are isomorphic ([LS]). Denote this line bundle on $\mathcal{M}_{\mathrm{Spin}(r)}$ by $\mathcal{P}$. The line bundle $\mathcal{P}$ is the positive generator of the Picard group of the stack $\mathcal{M}_{\mathrm{Spin}(r)}$. It comes equipped with sections s_κ for each theta characteristic κ, coming from the identification $p^*\mathcal{P}_\kappa \xrightarrow{\sim} \mathcal{P}$ (s_κ are well defined up to scalars).

Let $\pi : \mathcal{X} \to S$ be a smooth projective relative curve. Assume by passing to an étale cover that the sheaf of theta-characteristics on the fibers of π is trivialized (as well as the sheaf of two torsion in the Jacobians of the fibers of π).

Question 6.1. *Do the sections s_κ form a projectively flat basis of $H^0(\mathcal{M}_{\mathrm{Spin}(r)}(\mathcal{X}_s), \mathcal{P})$?*

A positive answer to this question, together with Proposition 5.10, would imply that the symplectic strange duality considered in [Be] is projectively flat. This is because (see [LS]) the pull back of s_κ to the product of moduli spaces $M_{\mathrm{Sp}(2m)}(X_s) \times M_{\mathrm{Sp}(2n)}(X_s)$ has the zero locus (as a divisor) $\frac{1}{2}\Delta$ where

$$\Delta = \{(E, F) : h^0(E \otimes F \otimes \kappa) \neq 0\}.$$

Appendix A. Generalities on projective connections

Let V be a vector bundle on a complex analytic manifold S.

- A holomorphic connection on V is a map

  $$\nabla : V \to V \otimes_{\mathcal{O}_S} \Omega^1$$

  so that $\nabla(fv) = f\nabla(v) + v \otimes df$ for all functions f and sections v of V.

The difference of any two such connections $\nabla - \nabla'$ is function linear and hence an element of $\mathrm{Hom}(V, V \otimes \Omega^1)$. We will say that ∇ and ∇' are projectively equivalent if

$$\nabla - \nabla' = \mathrm{Id} \otimes \omega$$

for some 1 form ω.

- A projective connection on V is a collection $(U_i, \nabla(i))$ such that U_i form an open cover of S and $\nabla(i)$ a connection on V restricted to U_i, along with the condition that $\nabla(i)$ and $\nabla(j)$ are projectively equivalent on $U_i \cap U_j$.

Suggestively,

$$\nabla(i)_Y v - \nabla(j)_Y v = \omega_{i,j}(Y)v$$

for all vector fields Y and indices i and j. Here $\omega_{i,j}$ is a 1-form on $U_i \cap U_j$. Therefore we can make sense of $\bar{\nabla} v$ as an element of $(V/\mathbb{C}v) \otimes \Omega^1$.

- A map $T : (V, \nabla) \to (W, \nabla')$ preserves projective connections if $\nabla'(Tv) - T(\nabla v) = T(v) \otimes \omega$ for some 1-form ω (these are local conditions).
- A section v of V is projectively flat if $\nabla v = v \otimes \omega$ for some 1-form ω.

The trivial bundle has an obvious projective connection. The projective flatness of v is clearly equivalent to: The map $\mathcal{O} \to V, 1 \mapsto v$ preserves projective connections.

If ∇ and ∇' are connections on V and W, then there is a connection $\tilde{\nabla}$ on $V \otimes_{\mathcal{O}} W$. This starts life as follows

$$\tilde{\nabla}(v, w) = \nabla v \otimes w + v \otimes \nabla' w$$

clearly $\tilde{\nabla}(fv, w) = \tilde{\nabla}(v, fw) = f\tilde{\nabla}(v, w) + df v \otimes w$, therefore $\tilde{\nabla}$ gives a connection on $V \otimes_{\mathcal{O}} W$. If we replace ∇ by something projectively equivalent to it, then the resulting $\tilde{\nabla}$ is projectively equivalent to the old one. *Therefore the tensor product of projective connections is well defined.*

The dual ∇^* of an ordinary connection ∇ on V is defined by

$$d\langle v, v^* \rangle = \langle \nabla v, v^* \rangle + \langle v, \nabla^* v^* \rangle$$

Therefore if ∇ and ∇' are projectively equivalent

$$\nabla - \nabla' = Id \otimes \omega,$$

then

$$\langle v, v^* \rangle \otimes \omega + \langle v, (\nabla^* - \nabla'^*) v^* \rangle = 0.$$

Hence one concludes that $\nabla^* - \nabla'^* = -Id \otimes \omega$. *Therefore the dual of a projective connection is well defined.*

Lemma A.1. *Let $T : (V, \nabla) \to (W, \nabla')$ be a projectively flat map of vector bundles with projective connections. Then the rank of T is locally constant.*

Proof. We can immediately reduce to the case of S a small open neighborhood of 0 in $\mathbb{C}$ and ∇, ∇' trivial connections on the trivial bundles V and W. Let $T(e_i) = (\sum \lambda_{ij}(t) f_j) \otimes dt$.

Define f from $\nabla'(Tv) - T(\nabla v) = T(v) \otimes f dt$. So we have $\frac{d}{dt}\lambda_{i,j}(t) = f(t)\lambda_{i,j}$. Let g be an antiderivative of f with $g(0) = 0$. Then

$$\lambda_{i,j}(t) = C_{i,j} e^{g(t)}$$

for all i, j where $C_{i,j}$ are constants. Hence the determinants of the minor of the matrix T in the basis e_i, f_j are constant up to exponential factors. ☐

Lemma A.2. *Let V, W be vector bundles with projective connections on S and s a projectively flat section of $V \otimes W$. Then the resulting map $\tilde{s} : V^* \to W$ is projectively flat. Conversely, if $\tilde{s}$ is projectively flat, then s is a projectively flat section.*

Proof. Write $s = \sum \theta_{ij} v_i \otimes w_j$, $\nabla v_i = \sum \lambda_{ia} v_a$ and $\nabla w_j = \sum \mu_{jb} w_b$.

We know $\nabla s = s\omega$ for some 1-form ω. This gives

$$\sum_{i,j} \theta_{ij} (\sum_a \lambda_{ia} v_a \otimes w_j + \sum_b \mu_{jb} v_i \otimes w_b) + \sum_{ij} d\theta_{ij} v_i \otimes w_j = \omega \sum_{ij} \theta_{ij} v_i \otimes w_j$$

Collecting coefficients of $v_a \otimes w_b$ we get

$$\sum_i \theta_{ib} \lambda_{ia} + \sum_j \theta_{aj} \mu_{jb} + d(\theta_{ab}) = \theta_{ab} \omega$$

We compute that $\tilde{s}(v_a^*) = \sum \theta_{aj} w_j$ where $v_a^* \in V^*$ form a basis dual to the basis v_a of V.. Therefore

$$\begin{aligned} \nabla \tilde{s}(v_a^*) &= \sum_j d\theta_{aj} w_j + \sum_{j,b} \theta_{aj} \mu_{jb} w_b \\ &= \sum_b (d\theta_{ab} + \sum_j \theta_{aj} \mu_{jb}) w_b \end{aligned}$$

On the other hand, $\nabla v_a^* = -\sum_i \lambda_{ia} v_i^*$. Hence

$$\tilde{s}(\nabla v_a^*) = -\sum_{i,b} \theta_{ib} \lambda_{i,a} w_b$$

Putting these together,

$$\begin{aligned} \nabla \tilde{s}(v_a^*) - \tilde{s}(\nabla v_a^*) &= \sum_b (d\theta_{ab} + \sum_j \theta_{aj} \mu_{jb} + \sum_i \theta_{i,b} \lambda_{i,a}) w_b \\ &= \sum_b \theta_{ab} w_b \omega = \tilde{s}(v_a^*) \omega \end{aligned}$$

We omit the (now easy) other direction. This part is not used in the paper. □

Lemma A.3. *Let G be a group of automorphisms of a vector bundle V on a space S (G acts trivially on S) with a projective connection ∇. Assume that G preserves ∇ projectively, $V^G \neq 0$, and some power of every $g \in G$ acts as a scalar (which must be 1, because there are invariants). Then, ∇ preserves the subsheaf $V^G \subseteq V$.*

Proof. Let v be a section of V over a sufficiently small open subset U of S. We have $g(\nabla_Y v) = \nabla_Y g(v) + \omega_g(Y) g v$ for some 1-form ω_g on U. If $v \in V^G$, then $g(\nabla_Y v) = \nabla_Y v + \omega_g(Y) v$, so for $k > 0$

$$g^k(\nabla_Y v) = \nabla_Y v + k\omega_g(Y) v.$$

If we pick k so that g^k as an endomorphism of V is the identity, we find that $\omega_g(Y)v = 0$ and hence $\nabla_Y v \in V^G$. □

Remark A.4. *Note that if a reductive group acts on a vector bundle V over a scheme S, $V^G \subseteq V$ is a subbundle whose fiber over any $s \in S$ is $(V_s)^G$.*

Proposition A.5. *Let V be a vector bundle on a space S, and suppose that ∇ and ∇' are connections on the vector bundle V, with the following property: For every $s \in S$, any tangent vector Y at s and any local section v of V in a neighborhood of s such that $(\nabla_Y v)(s) = 0$, we have $(\nabla'_X v)(s) = c(X, Y, v) v(s)$ for some $c(X, Y, v) \in \mathbb{C}$. Then ∇ and ∇' are projectively equivalent.*

Proof. Clearly, $c(X,Y,v)$ depends just on the point s and the vector field Y and not upon v (by taking sums and differences of the v's). The difference $(\nabla_Y - \nabla'_Y)$ is function linear as an operator on V (and also in Y), and to find its value at (Y,s), it suffices to evaluate on sections v such that $\nabla_Y(v) = 0$. $\square$

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